Characterisations of elementary pseudo-caps and good eggs

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Abstract

In this note, we use the theory of Desarguesian spreads to investigate good eggs. Thas showed that an egg in $\mathrm{PG}(4n-1,q)$, q odd, with two good elements is elementary. By a short combinatorial argument, we show that a similar statement holds for large pseudo-caps, in odd and even characteristic. As a corollary, this improves and extends the result of Thas, Thas and Van Maldeghem (2006) where one needs at least 4 good elements of an egg in even characteristic to obtain the same conclusion. We rephrase this corollary to obtain a characterisation of the generalised quadrangle $T_3(\mathcal{O})$ of Tits.

Lavrauw (2005) characterises elementary eggs in odd characteristic as those good eggs containing a space that contains at least 5 elements of the egg, but not the good element. We provide an adaptation of this characterisation for weak eggs in odd and even characteristic. As a corollary, we obtain a direct geometric proof for the theorem of Lavrauw.

1 Preliminaries

In this note, we study *eggs* and *pseudo-caps* in the projective space $\mathrm{PG}(n,q)$, where $\mathrm{PG}(n,q)$ denotes the n-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, $q=p^h$, p prime. Many previous proofs and characterisations of eggs rely on the connection with eggs and *translation generalised quadrangles* [16]. It is our aim to study eggs from a purely geometric perspective, without using this connection or coordinates. In Section 2 we obtain a connection between good eggs and Desarguesian spreads. This link will enable us to reprove, improve or extend known results in Sections 3 and 4. We begin by repeating some well-known definitions.

Definition. A *cap* in $\mathrm{PG}(n,q)$ is a set of points such that every three points span a plane. A cap of size k is denoted as a *k-cap*.

A k-cap of $\mathrm{PG}(2,q)$ is often called a *k-arc*. A k-arc in $\mathrm{PG}(2,q)$ satisfies $k\leq q+1$ for q odd and $k\leq q+2$ for q even. A $(q+1)$-arc is called an *oval*, a $(q+2)$-arc a *hyperoval*. A k-cap of $\mathrm{PG}(3,q)$, $q>2$ satisfies $k\leq q^2+1$, moreover, a (q^2+1)-cap of $\mathrm{PG}(3,q)$ is often called an *ovoid*. We will consider the higher dimensional equivalent of these structures.

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Definition. A *pseudo-cap* is a set $\mathcal{C}$ of $(n-1)$-spaces in $\mathrm{PG}(2n+m-1,q)$ such that any three elements of $\mathcal{C}$ span a $(3n-1)$-space.

If $m = n$, a pseudo-cap is often called a *pseudo-arc*. By [15], a pseudo-arc $\mathcal{A}$ in $\mathrm{PG}(3n-1,q)$ satisfies $|\mathcal{A}| \leq q^n+1$ for q odd and $|\mathcal{A}| \leq q^n+2$ for q even. If a pseudo-arc $\mathcal{A}$ has q^n+1 or q^n+2 elements, $\mathcal{A}$ is a *pseudo-oval* or *pseudo-hyperoval* respectively. If $m = 2n$, a pseudo-cap with $q^{2n}+1$ elements is called a *pseudo-ovoid*.

Examples of pseudo-caps in $\mathrm{PG}(kn-1,q)$ arise by applying field reduction to caps in $\mathrm{PG}(k-1,q^n)$ and if a pseudo-cap is obtained by field reduction, we call it *elementary*. *Field reduction* is the concept where a point in $\mathrm{PG}(k-1,q^n)$ corresponds in a natural way to an $(n-1)$-space of $\mathrm{PG}(kn-1,q)$. The set of all points of $\mathrm{PG}(k-1,q^n)$ then correspond to a set of disjoint $(n-1)$-spaces partitioning $\mathrm{PG}(kn-1,q)$, forming a *Desarguesian spread*. Every Desarguesian spread $\mathcal{D}$ has the property that the space spanned by 2 elements of $\mathcal{D}$ is partitioned by elements of $\mathcal{D}$, i.e. $\mathcal{D}$ is *normal*. Moreover, a normal $(n-1)$-spread of $\mathrm{PG}(kn-1,q)$, $k > 2$, is Desarguesian, see [2]. For more information on field reduction and Desarguesian spreads we refer to [9].

A *partial spread* in $\mathrm{PG}(n+m-1,q)$ is a set of mutually disjoint $(n-1)$-spaces. Every element E_i of a pseudo-cap $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,q)$ defines a partial spread

$$\mathcal{S}_i := \{E_0, \ldots, E_{i-1}, E_{i+1}, \ldots, E_{|\mathcal{E}|}\}/E_i$$

in $\mathrm{PG}(n+m-1,q) \cong \mathrm{PG}(2n+m-1,q)/E_i$ and we say that the element E_i *induces* the partial spread $\mathcal{S}_i$.

A partial spread of $\mathrm{PG}(2n-1,q)$ of size q^n is said to have *deficiency 1*. From [3], we know that a partial spread of $\mathrm{PG}(2n-1,q)$ with deficiency 1 can be extended to a spread in a unique way, i.e. the set of points in $\mathrm{PG}(2n-1,q)$ not lying on an element of the partial spread, form an $(n-1)$-space.

Definition. A *weak egg* in $\mathrm{PG}(2n+m-1,q)$ is a pseudo-cap of size q^m+1.

Clearly, pseudo-ovals and pseudo-ovoids are examples of weak eggs. A weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is called an *egg* if each element $E \in \mathcal{E}$ is contained in a $(n+m-1)$-space, T_E, which is skew from every element of $\mathcal{E}$ different from E. The space T_E is called the *tangent space* of $\mathcal{E}$ at E. It is not hard to show that if $n = m$, then every weak egg is an egg. Eggs are studied mostly because of their one-to-one correspondance with *translation generalised quadrangles* of order (q^n, q^{2n}), see Subsection 3.2.

The only known examples of eggs in $\mathrm{PG}(2n+m-1,q)$ have either $m = n$ or $m = 2n$ and we have the following theorem restricting the number of possibilities for the parameters n and m.

Theorem 1.1. [11, Theorem 8.7.2] *If $\mathcal{E}$ is an egg of $\mathrm{PG}(2n+m-1,q)$ then $m = n$ or $ma = n(a+1)$ with a odd. Moreover, if q even, then $m = n$ or $m = 2n$.*

This explains why the study of eggs is mainly focused on pseudo-ovals and pseudo-ovoids.

In the case of pseudo-ovals, all known examples are elementary. The classical example of an oval in $\mathrm{PG}(2,q^n)$ is a conic. It is a well-known theorem of Segre that an oval of $\mathrm{PG}(2,q^n)$, q odd, is always a conic. A *pseudo-conic* in $\mathrm{PG}(3n-1,q)$ is an elementary

pseudo-oval, arising from applying field reduction to a conic in $\mathrm{PG}(2,q^n)$. We have the following theorems characterising elementary pseudo-ovals using the induced Desarguesian spreads.

Theorem 1.2. [6] *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1,q)$, q odd, such that for at least one element the induced spread is Desarguesian, then $\mathcal{O}$ is a pseudo-conic.*

Theorem 1.3. [13] *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1,q)$, n prime, $q>2$ even, such that all elements induce a Desarguesian spread, then $\mathcal{O}$ is elementary.*

In the case that q is odd, we have the following theorem which extends Theorem 1.2 from pseudo-ovals to large pseudo-arcs in $\mathrm{PG}(3n-1,q)$.

Theorem 1.4. [12] *If $\mathcal{K}=\{K_1,\ldots,K_s\}$ is a pseudo-arc in $\mathrm{PG}(3n-1,q)$, q odd, of size greater than the size of the second largest complete arc in $\mathrm{PG}(2,q^n)$, where for one element K_i of $\mathcal{K}$, the partial spread $\mathcal{S}=\{K_1,\ldots,K_{i-1},K_{i+1},\ldots,K_s\}/K_i$ extends to a Desarguesian spread of $\mathrm{PG}(2n-1,q)=\mathrm{PG}(3n-1,q)/K_i$, then $\mathcal{K}$ is contained in a pseudo-conic.*

In Theorem 3.5, we will prove a similar statement for pseudo-caps in $\mathrm{PG}(4n-1,q)$.

All known examples of pseudo-ovoids in $\mathrm{PG}(4n-1,q)$ are elementary when q is even, but in contrast to the situation for pseudo-ovals, when q is odd, there are non-elementary examples of pseudo-ovoids. The standard example of an ovoid in $\mathrm{PG}(3,q^n)$ is an elliptic quadric $Q^-(3,q^n)$. By the famous result of Barlotti and Panella [1, 10], every ovoid of $\mathrm{PG}(3,q^n)$, q odd, is an elliptic quadric $Q^-(3,q^n)$, however, there is no classification of ovoids in $\mathrm{PG}(3,q^n)$ for q even. For both even and odd order q, the classification of pseudo-ovoids is an open problem.

2 Good eggs and Desarguesian spreads

A (weak) egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$, $m>n$, is *good* at an element $E\in\mathcal{E}$ if every $(3n-1)$-space containing E and at least two other elements of $\mathcal{E}$, contains exactly q^n+1 elements of $\mathcal{E}$. A (weak) egg that has at least one good element is called a *good (weak) egg*. If $\mathcal{E}$ is good at E, then for any two elements $E_1,E_2\in\mathcal{E}\backslash\{E\}$ the $(3n-1)$-space $\langle E,E_1,E_2\rangle$ intersects $\mathcal{E}$ in a pseudo-oval.

Lemma 2.1. *Good weak eggs in $\mathrm{PG}(2n+m-1,q)$ can only exist if n is a divisor of m. Good eggs only exist in $\mathrm{PG}(4n-1,q)$.*

Proof. Consider a weak egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,q)$, $m>n$, good at an element $E_1\in\mathcal{E}$. Consider a second element $E_2\in\mathcal{E}\backslash\{E_1\}$. For every element $E\in\mathcal{E}\setminus\{E_1,E_2\}$, the $(3n-1)$-space $\langle E,E_1,E_2\rangle$ intersects $\mathcal{E}$ in a pseudo-oval. By considering the elements of $\mathcal{E}\setminus\{E_1,E_2\}$, we find a set $\mathcal{T}$ of $(3n-1)$-spaces containing $\langle E_1,E_2\rangle$, such that each space of $\mathcal{T}$ intersects $\mathcal{E}$ in a pseudo-oval. Every two spaces in $\mathcal{T}$ meet exactly in $\langle E_1,E_2\rangle$ and $\mathcal{E}$ is the union of the pseudo-ovals $\{T\cap\mathcal{E}|T\in\mathcal{T}\}$. The set $\mathcal{T}$ contains $\frac{q^m-1}{q^n-1}$ $(3n-1)$-spaces; as q^n-1 has to be a divisor of q^m-1, it follows that n is a divisor of m.

Suppose $\mathcal{E}$ is an egg. For q even, by Theorem 1.1, eggs only exist in $\mathrm{PG}(4n-1,q)$ (or $\mathrm{PG}(3n-1,q)$). Consider now a good egg of $\mathrm{PG}(2n+m-1,q)$, q odd, where m is a multiple of n. By Theorem 1.1, $m=\frac{a+1}{a}n$, for some odd integer a, so we conclude that $m=2n$. $\square$

We will show that the good elements of an egg are exactly those inducing a partial spread which is extendable to a Desarguesian spread. Part (i) of the following theorem, for $\mathcal{E}$ an egg, is mentioned in [16, Remark 5.1.7].

Theorem 2.2.

(i) *If a weak egg* $\mathcal{E}$ *in* $\mathrm{PG}(2n+m-1,q)$ *is good at an element* E*, then* E *induces a partial spread which extends to a Desarguesian spread.*

(ii) *Let* $\mathcal{E}$ *be a weak egg in* $\mathrm{PG}(2n+m-1,q)$ *for* q *odd and an egg in* $\mathrm{PG}(2n+m-1,q)$ *for* q *even. If an element* $E \in \mathcal{E}$ *induces a partial spread which extends to a Desarguesian spread, then* $\mathcal{E}$ *is good at* E.

Proof. (i) Suppose $\mathcal{E}$ is a weak egg which is good at E. Consider the partial spread $\mathcal{S}$ of $\mathrm{PG}(n+m-1,q)$ of size q^m induced by E. Because $\mathcal{E}$ is good at E, any two elements of $\mathcal{S}$ span a $(2n-1)$-space which contains a partial spread of q^n elements of $\mathcal{S}$. This partial spread has deficiency 1, so extends uniquely to a spread by one $(n-1)$-space (by [3]).

Consider three elements $S_1, S_2, S_3 \in \mathcal{S}$ not lying in the same $(2n-1)$-space, hence spanning a $(3n-1)$-space π. There are q^n elements of $\mathcal{S}$ contained in $\langle S_2, S_3\rangle$. For every element R of $\mathcal{S} \cap \langle S_2, S_3\rangle$, the $(2n-1)$-space $\langle S_1, R\rangle$ contains q^n elements of $\mathcal{S}$. Hence, there are q^n $(2n-1)$-spaces of π containing S_1 and q^n-1 other elements of $\mathcal{S}$. Similarly, there are q^n $(2n-1)$-spaces of π containing S_2 and q^n-1 other elements of $\mathcal{S}$. Since π has dimension $3n-1$, two such distinct $(2n-1)$-spaces, one containing S_1 and the other containing S_2, intersect in at least an $(n-1)$-space, hence, in exactly an $(n-1)$-space. This space is either an element of $\mathcal{S}$ or the $(n-1)$-space which extends both of them to a spread. It follows that there are q^{2n} elements of $\mathcal{S}$ contained in π and if an element of $\mathcal{S}$ intersects π, then it is contained in π. Hence, if $\langle S_2, S_3\rangle$ meets a $(2n-1)$-space spanned by S_1 and an other element of $\mathcal{S}$, then they meet in an $(n-1)$-space.

As S_1, S_2, S_3 were chosen randomly, it follows in general that if two distinct $(2n-1)$-spaces spanned by elements of $\mathcal{S}$ intersect, then they meet in an $(n-1)$-space. They meet either in an $(n-1)$-space of $\mathcal{S}$ or in the $(n-1)$-space which extends the partial spreads of both $(2n-1)$-spaces to a spread. We see that $\mathcal{S}$ can be uniquely extended to a spread which is normal, thus Desarguesian.

(ii) Now let $\mathcal{E}$ be an egg if q is even and a weak egg if q is odd. Suppose E induces a partial spread $\mathcal{S}$ of size q^m which extends to a Desarguesian $(n-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(n+m-1,q)$, hence $m = kn$ for some $k > 1$. There are $\frac{q^m-1}{q^n-1}$ elements of $\mathcal{D}$ not contained in $\mathcal{S}$.

When $\mathcal{E}$ is an egg, the elements of $\mathcal{D}\backslash\mathcal{S}$ span a $(m-1)$-space, corresponding to T_E. Hence, any $(2n-1)$-space spanned by two elements of $\mathcal{S}$ contains q^n elements of $\mathcal{S}$ and one element $\mathcal{D}\backslash\mathcal{S}$. So, $\mathcal{E}$ is good at E.

Suppose $\mathcal{E}$ is a weak egg, with q odd. As q is odd, no $(3n-1)$-space intersects $\mathcal{E}$ in a pseudo-hyperoval. Hence, any $(3n-1)$-space containing E intersects $\mathcal{E}$ in at most q^n+1 elements, so any $(2n-1)$-space spanned by two elements of $\mathcal{S}$ can contain at most q^n elements of $\mathcal{S}$. Hence, any such space must contain at least one element of $\mathcal{D}\backslash\mathcal{S}$. By field reduction, the elements of the Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(n+m-1,q)$ are in one-to-one correspondance with the points of $\mathrm{PG}(\frac{m}{n}, q^n)$. Any $(2n-1)$-space spanned by two elements of $\mathcal{D}$ must contain at least one element of $\mathcal{D}\backslash\mathcal{S}$. Hence, the points

corresponding to $\mathcal{D}\backslash\mathcal{S}$ form a line-blocking set of $\mathrm{PG}(\frac{m}{n}, q^n)$. Since $|\mathcal{D}\backslash\mathcal{S}| = \frac{q^m-1}{q^n-1}$, from [4] it follows that the points corresponding to $\mathcal{D}\backslash\mathcal{S}$ are the points of a $(\frac{m}{n}-1)$-space, hence the elements of $\mathcal{D}\backslash\mathcal{S}$ span a $(m-1)$-space. As before, it follows that $\mathcal{E}$ is good at E. □

The following corollary, for $\mathcal{E}$ an egg, was also mentioned in [14, Theorem 4.3.4] in terms of translation generalised quadrangles.

Corollary 2.3. *If a weak egg $\mathcal{E}$, q odd, is good at an element E, then every pseudo-oval of $\mathcal{E}$ containing E is a pseudo-conic.*

Proof. Let Π be a $(n+m-1)$-space disjoint from E. By Theorem 2.2, the partial spread $\mathcal{E}/E$ in Π extends to a Desarguesian spread. Consider a pseudo-oval $\mathcal{O}$ of $\mathcal{E}$ containing E. The q^n elements of $\mathcal{O}/E$ are contained in $\mathcal{E}/E$ and thus extend to a Desarguesian spread of the $(2n-1)$-space $\langle\mathcal{O}\rangle \cap \Pi$.

The element E of the pseudo-oval $\mathcal{O}$ induces a partial spread $\mathcal{O}/E$ which extends to a Desarguesian spread, hence, by Theorem 1.2, the statement follows. □

3 Characterising good eggs and translation generalised quadrangles of order (q^n, q^{2n})

3.1 Eggs with two good elements

An elementary pseudo-ovoid that arises from applying field reduction to an elliptic quadric is called *classical*. We recall the following theorem from [16].

Theorem 3.1. [16, Theorem 5.1.12]

If q is odd and an egg $\mathcal{E}$ in $\mathrm{PG}(4n-1, q)$ has at least two good elements, then $\mathcal{E}$ is classical. If q is even and an egg $\mathcal{E}$ in $\mathrm{PG}(4n-1, q)$ has at least four good elements, not contained in a common pseudo-oval on $\mathcal{E}$, then $\mathcal{E}$ is elementary.

It was open problem whether, for q even, being good at two elements is sufficient to be elementary, this was posed as Problem A.5.6 in [16]. We will give an affirmative answer to this question in a more general setting, namely in terms of pseudo-caps. We first need two lemma's concerning Desarguesian spreads.

Lemma 3.2. [13, Corollary 1.8] *Consider two Desarguesian $(n-1)$-spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathrm{PG}(2n-1, q)$, $q > 2$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ have at least 3 elements in common, then they share exactly $q^t + 1$ elements for some $t|n$.*

The following lemma is a generalisation of [13, Lemma 1.4] and the proof is analogous. We introduce some necessary definitions and notations.

A *regulus* $\mathcal{R}$ in $\mathrm{PG}(2n-1, q)$ is a set of $q+1$ mutually disjoint $(n-1)$-spaces having the property that if a line meets 3 elements of $\mathcal{R}$, then it meets all elements of $\mathcal{R}$. Let us denote the unique regulus through 3 mutually disjoint $(n-1)$-spaces A, B and C in $\mathrm{PG}(2n-1, q)$ by $\mathcal{R}(A, B, C)$. Every Desarguesian spread $\mathcal{D}$ has the property that for 3 elements A, B, C in $\mathcal{D}$, the elements of $\mathcal{R}(A, B, C)$ are also contained in $\mathcal{D}$, i.e. $\mathcal{D}$ is regular (see also [5]).

We will use the following notation for points of a projective space $\mathrm{PG}(r-1,q^n)$. A point P of $\mathrm{PG}(r-1,q^n)$ defined by a vector $(x_1,x_2,\dots,x_r)\in(\mathbb{F}_{q^n})^r$ is denoted by $\mathbb{F}_{q^n}(x_1,x_2,\dots,x_r)$, reflecting the fact that every $\mathbb{F}_{q^n}$-multiple of $(x_1,x_2,\dots,x_r)$ gives rise to the point P. We can identify the vector space $\mathbb{F}_{q^{nr}}$ with $(\mathbb{F}_{q^n})^r$, and hence write every point of $\mathrm{PG}(rn-1,q)$ as $\mathbb{F}_q(x_1,\dots,x_r)$ with $x_i\in\mathbb{F}_{q^n}$. In this way, by field reduction, a point $\mathbb{F}_{q^n}(x_1,\dots,x_r)$ in $\mathrm{PG}(r-1,q^n)$ corresponds to the $(n-1)$-space $\mathbb{F}_{q^n}(x_1,\dots,x_r)=\{\mathbb{F}_q(\alpha x_1,\dots,\alpha x_r)|\alpha\in\mathbb{F}_{q^n}\}$ of $\mathrm{PG}(rn-1,q)$.

Lemma 3.3. *Let $\mathcal{D}_1$ be a Desarguesian $(n-1)$-spread in a $(kn-1)$-dimensional subspace Π of $\mathrm{PG}((k+1)n-1,q)$, let μ be an element of $\mathcal{D}_1$ and let E_1 and E_2 be mutually disjoint $(n-1)$-spaces such that $\langle E_1,E_2\rangle$ meets Π exactly in the space μ. Then there exists a unique Desarguesian $(n-1)$-spread of $\mathrm{PG}((k+1)n-1,q)$ containing E_1, E_2 and all elements of $\mathcal{D}_1$.*

Proof. Since $\mathcal{D}_1$ is a Desarguesian spread in Π, we can choose coordinates for Π such that $\mathcal{D}_1=\{\mathbb{F}_{q^n}(x_1,x_2,\dots,x_k)|x_i\in\mathbb{F}_{q^n}\}$ and $\mu=\mathbb{F}_{q^n}(0,\dots,0,1)$. We embed Π in $\mathrm{PG}((k+1)n-1,q)$ by mapping a point $\mathbb{F}_q(x_1,\dots,x_k)$, $x_i\in\mathbb{F}_{q^n}$, of Π onto $\mathbb{F}_q(x_1,\dots,x_k,0)$. Let ℓ_P denote the unique transversal line through a point P of μ to the regulus $\mathcal{R}(\mu,E_1,E_2)$.

We can still choose coordinates for $n+1$ points in general position in $\mathrm{PG}((k+1)n-1,q)\setminus\Pi$. We will choose these $n+1$ points such that n of them belong to E_1 and one of them belongs to E_2. Consider a set $\{y_i|i=1,\dots,n\}$ forming a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. We may assume that the line ℓ_{P_i} through $P_i=\mathbb{F}_q(0,\dots,0,y_i,0)\in\mu$ meets E_1 in the point $\mathbb{F}_q(0,\dots,0,0,y_i)$. It follows that $E_1=\mathbb{F}_{q^n}(0,\dots,0,0,1)$. Moreover, we may assume that ℓ_Q with $Q=\mathbb{F}_q(0,\dots,0,0,\sum_{i=1}^n y_i,0)\in\mu$ meets E_2 in $\mathbb{F}_q(0,\dots,0,\sum_{i=1}^n y_i,\sum_{i=1}^n y_i)$. Since $\mathbb{F}_q(0,\dots,0\sum_{i=1}^n y_i,\sum_{i=1}^n y_i)$ has to be in the space spanned by the intersection points $R_i=\ell_{P_i}\cap E_2$, it follows that $R_i=\mathbb{F}_q(0,\dots,0,y_i,y_i)$ and consequently, that $E_2=\mathbb{F}_{q^n}(0,\dots,0,1,1)$.

It is clear that the Desarguesian spread $\mathcal{D}=\{\mathbb{F}_{q^n}(x_1,\dots,x_{k+1})|x_i\in\mathbb{F}_{q^n}\}$ contains the spread $\mathcal{D}_1$ and the $(n-1)$-spaces E_1 and E_2. Moreover, since every element of $\mathcal{D}$, not in $\langle E_1,E_2\rangle$, is obtained as the intersection of $\langle E_1,X\rangle\cap\langle E_2,Y\rangle$, where $X,Y\in\mathcal{D}_1$, it is clear that $\mathcal{D}$ is the unique Desarguesian spread satisfying our hypothesis. ☐

Lemma 3.4. *Consider a pseudo-cap $\mathcal{E}$ of $\mathrm{PG}(4n-1,q)$ containing an element E that induces a partial spread which extends to a Desarguesian spread. If Π is a $(3n-1)$-space spanned by E and two other elements of $\mathcal{E}$, then every element of $\mathcal{E}$ is either disjoint from Π or contained in Π.*

Proof. Let Σ be a $(3n-1)$-space skew from E and consider the induced partial spread $\mathcal{E}/E$ in Σ. If F is an element of $\mathcal{E}$ which meets Π, then the projection F/E of F from E onto Σ is an element of $\mathcal{E}/E$ which meets the space Π/E. By assumption, the space Π/E is spanned by spread elements of a partial spread extending to a Desarguesian spread. Hence, since a Desarguesian spread is normal, F/E is contained in Π/E. It follows that since Π contains E, the element F is contained in Π. ☐

Theorem 3.5. *Consider a pseudo-cap $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$, $q>2$, with $|\mathcal{E}|>q^{n+k}+q^n-q^k+1$, q odd, and $|\mathcal{E}|>q^{n+k}+q^n+2$, q even, where k is the largest divisor of n with $k\neq n$. The pseudo-cap $\mathcal{E}$ is elementary if and only if two of its elements induce a partial spread which extends to a Desarguesian spread.*

Proof. If $\mathcal{E}$ is elementary, then the elements of $\mathcal{E}$ are contained in a Desarguesian spread of $\mathrm{PG}(4n-1,q)$, so every element of $\mathcal{E}$ induces a partial spread which extends to a Desarguesian spread.

Now suppose that $\mathcal{E}$ contains two distinct elements E_1, E_2 that induce a partial spread which extends to a Desarguesian spread. Since $|\mathcal{E}| > q^n + 2$, using Lemma 3.4, we can find two elements $E_3, E_4 \in \mathcal{E}$ such that $\langle E_1, E_2, E_3, E_4\rangle$ spans $\mathrm{PG}(4n-1,q)$.

The partial spread induced by E_1 in the space $\langle E_2, E_3, E_4\rangle$ can be extended to a Desarguesian spread $\mathcal{D}_1$. Analogously, the partial spread induced by E_2 in the space $\langle E_1, E_3, E_4\rangle$ can be extended to a Desarguesian spread $\mathcal{D}_2$. Since E_3 and E_4 are elements of the spreads $\mathcal{D}_1$ and $\mathcal{D}_2$, the Desarguesian spreads $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect the $(2n-1)$-space $\langle E_3, E_4\rangle$ each in a Desarguesian spread, say $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.

Take an element $E \in \mathcal{E}\backslash\{E_1, E_2\}$ and consider the $(3n-1)$-space $\langle E_1, E_2, E\rangle$. From Lemma 3.4 it follows that any element of $\mathcal{E}$ is either contained in or disjoint from $\langle E_1, E_2, E\rangle$. By considering the elements of $\mathcal{E}\setminus\{E_1, E_2\}$, we find a set $\mathcal{T}$ of $(3n-1)$-spaces containing $\langle E_1, E_2\rangle$, such that each space of $\mathcal{T}$ intersects $\mathcal{E}$ in a pseudo-arc. Every two spaces in $\mathcal{T}$ meet exactly in $\langle E_1, E_2\rangle$ and $\mathcal{E}$ is the union of the pseudo-arcs $\{T\cap\mathcal{E}|T \in \mathcal{T}\}$. The set $\mathcal{T}$ intersects $\langle E_3, E_4\rangle$ in a partial $(n-1)$-spread $\mathcal{P}$.

Let P be an element of $\mathcal{P}$, then $\langle P, E_1, E_2\rangle$ is a $(3n-1)$-space containing at least one element E of $\mathcal{E}\backslash\{E_1, E_2\}$. The projection E' of E from E_1 onto $\langle E_2, E_3, E_4\rangle$ is contained in $\mathcal{D}_1$. We obtain that $P = \langle E_2, E'\rangle \cap \langle E_3, E_4\rangle$, and since the elements E_2, E', E_3, E_4 are contained in $\mathcal{D}_1$, this implies that P is contained in $\mathcal{D}_1$. Moreover, since $P \subset \langle E_3, E_4\rangle$, the element P is contained in $\mathcal{S}_1$. Similarly, we obtain that P is contained in $\mathcal{S}_2$ and we conclude that every element of $\mathcal{P}$ must be contained in both $\mathcal{S}_1$ and $\mathcal{S}_2$.

Suppose that k is the largest divisor of n with $k \neq n$. The pseudo-cap $\mathcal{E}$ has size $|\mathcal{E}| > (q^n - \epsilon)(q^k + 1) + 2$ and every $(3n-1)$-space of $\mathcal{T}$ contains at most $q^n - \epsilon$ elements different from E_1, E_2, where $\epsilon = 1$ for q odd and $\epsilon = 0$ for q even. By the pigeonhole principle, it follows that $|\mathcal{P}| \geq q^k + 2$. Hence, the Desarguesian spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ have at least $q^k + 2$ elements in common, where k is the largest divisor of n with $k \neq n$. As $q > 2$, by Lemma 3.2, we find that $\mathcal{S}_1 = \mathcal{S}_2$.

By Theorem 3.3, consider the unique Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(4n-1,q)$ containing all elements of $\mathcal{D}_1$ and two distinct elements of $\mathcal{D}_2\backslash\mathcal{D}_1$. It is clear that, since $\mathcal{S}_1 = \mathcal{S}_2$, the spread $\mathcal{D}$ contains all elements of $\mathcal{D}_2$.

Every element of $\mathcal{E}$, not in $\mathcal{D}_1 \cup \mathcal{D}_2$, arises as the intersection $\langle E_1, X\rangle \cap \langle E_2, Y\rangle$ for some $X \in \mathcal{D}_1 \subset \mathcal{D}$ and $Y \in \mathcal{D}_2 \subset \mathcal{D}$, hence, since a Desarguesian spread is normal, every element of $\mathcal{E}$ belongs to $\mathcal{D}$. It follows that $\mathcal{E}$ is elementary. $\square$

We obtain the following corollary which improves [16, Theorem 5.1.12].

Corollary 3.6. *A weak egg in* $\mathrm{PG}(4n-1,q)$ *which is good at two distinct elements is elementary.*

Proof. A weak egg is a pseudo-cap of size $q^{2n} + 1$ in $\mathrm{PG}(4n-1,q)$. By Theorem 2.2, if the weak egg is good at two elements, these elements induce a partial spread which extends to a Desarguesian spread. We can repeat the proof of Theorem 3.5. Now the partial spread $\mathcal{P}$ has size $q^n + 1$, so the conclusion $\mathcal{S}_1 = \mathcal{S}_2$ follows immediately. We do not require Lemma 3.2, hence the restriction $q > 2$ can be dropped. $\square$

3.2 A corollary in terms of translation generalised quadrangles

A *generalised quadrangle* of order (s,t), $s,t>1$, is an incidence structure of points and lines satisfying the following axioms:

- every line has exactly $s+1$ points,
- through every point, there are exactly $t+1$ lines,
- if P is a point, not on the line L, then there is exactly one line through P which meets L non-trivially.

From every *egg* $\mathcal{E}$ in $\Sigma_\infty = \mathrm{PG}(2n+m-1,q)$ we can construct a generalised quadrangle $(\mathcal{P},\mathcal{L})$ as follows. Embed Σ_∞ as a hyperplane at infinity of $\mathrm{PG}(2n+m,q)$.

$\mathcal{P}$: (i) *affine* points of $\mathrm{PG}(2n+m,q)$, i.e. the points not lying in Σ_∞,

(ii) the $(n+m)$-spaces meeting Σ_∞ in T_E for some $E\in\mathcal{E}$,

(iii) the symbol (∞).

$\mathcal{L}$: (a) the n-spaces meeting Σ_∞ in an element of $\mathcal{E}$,

(b) the elements of $\mathcal{E}$.

Incidence is defined as follows.

- A point of type (i) is incident with the lines of type (a) through it.
- A point of type (ii) is incident with the lines of type (a) it contains and the line of type (b) it contains.
- The point (∞) is incident with all lines of type (b).

The obtained generalised quadrangle is denoted as $T(\mathcal{E})$ and is called a *translation generalised quadrangle* (TGQ) with base point (∞). In [11, Theorem 8.7.1] it is proven that every TGQ of order (q^n,q^m), where $\mathbb{F}_q$ is a subfield of its kernel, is isomorphic to a $T(\mathcal{E})$ for some egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,q)$.

When $n=m=1$, then $\mathcal{O}$ is an oval of $\mathrm{PG}(2,q)$ and the construction above gives the well-known construction of $T_2(\mathcal{O})$. When $n=1$ and $m=2$, then $\mathcal{O}$ is an ovoid of $\mathrm{PG}(3,q)$ and the construction above is the construction of Tits of $T_3(\mathcal{O})$ (see [16]).

Lemma 3.7. *Let $T=T(\mathcal{E})$ be a TGQ of order (q^n,q^{2n}) with base point (∞). Let m_1,m_2,m_3 be three distinct lines through (∞), and let E_1, E_2, E_3 denote the elements of $\mathcal{E}$ corresponding to m_1,m_2,m_3 respectively. Then there is a subquadrangle of order q^n through m_1,m_2,m_3 if and only if the $(3n-1)$-dimensional space $\langle E_1,E_2,E_3\rangle$ contains exactly q^n+1 elements of $\mathcal{E}$.*

Proof. Suppose that the $(3n-1)$-space $\Sigma=\langle E_1,E_2,E_3\rangle$ contains a set $\mathcal{O}$ of exactly q^n+1 elements of $\mathcal{E}$, then it is clear that $T(\mathcal{E})$ defines the incidence structure $T(\mathcal{O})$ in a $3n$-space through Σ. The structure $T(\mathcal{O})$ is a generalised quadrangle of order q^n, forming a subquadrangle of $T(\mathcal{E})$ and containing the lines m_1,m_2,m_3.

On the other hand, suppose that there is a subquadrangle T' of order q^n containing m_1, m_2, m_3, where the lines m_1, m_2, m_3 are incident with (∞). This implies that the point (∞) is in T', and since (∞) lies only on lines of type (b) (i.e. the lines corresponding to elements of $\mathcal{E}$), we know that T' contains exactly $q^n + 1$ lines of type (b), among which the lines m_1, m_2 and m_3. Let $\{E_1, \ldots, E_{q^n+1}\}$ be the egg elements corresponding to these lines. This means that there are $(q^n + 1)q^{2n}$ lines in T' of type (a), containing in total $(q^n + 1)q^{2n}(q^n)/(q^n + 1) = q^{3n}$ points of type (i) (i.e. affine points).

Each $(n-1)$-space E_j is contained in q^{2n} n-spaces corresponding to a line of type (a) of T' and every affine point is contained in exactly one n-space containing E_j. Let P_j be a point of the space E_j, then we see that the q^{3n} affine points of T' lie on q^{2n} lines through P_j. As this holds for every $j \in \{1, \ldots, q^n + 1\}$, it is clear that the q^{3n} affine points of T' are contained in a $3n$-space. This in turn implies that the elements $\{E_1, \ldots, E_{q^n+1}\}$ are contained in a $(3n-1)$-space, namely $\langle E_1, E_2, E_3 \rangle$. Hence, this space contains at least $q^n + 1$ elements of $\mathcal{E}$. Since $\mathcal{E}$ is an egg, it is not possible that a $(3n-1)$-space contains more than $q^n + 1$ elements of $\mathcal{E}$, which concludes the proof. □

Lemma 3.8. *Let $T = T(\mathcal{E})$ be a TGQ of order (q^n, q^{2n}) with base point (∞). Let ℓ be a line through (∞) and E_ℓ the element of $\mathcal{E}$ corresponding to ℓ. The egg $\mathcal{E}$ is good at E_ℓ if and only if for every two distinct lines m_1, m_2 through (∞), where $m_1, m_2 \neq \ell$, there is a subquadrangle of order q^n through m_1, m_2, ℓ.*

Proof. This follows immediately from Lemma 3.7 and the definition of a being good at an element. □

We are now ready to state the promised characterisation of the translation generalised quadrangle $T_3(\mathcal{O})$.

Theorem 3.9. *Let T be a TGQ of order (q^n, q^{2n}) with base point (∞). Suppose that T contains two distinct lines ℓ_i, $i = 1, 2$ such that for every two distinct lines m_1, m_2 through (∞), where $m_1, m_2 \neq \ell_i$, $i = 1, 2$ there is a subquadrangle through m_1, m_2, ℓ_i, $i = 1, 2$, then T is isomorphic to $T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid of* $\mathrm{PG}(3, q^n)$.

4 A geometric proof of a Theorem of Lavrauw

In this section we obtain a second characterisation of good weak eggs. We need the following lemma stating that every good element of a weak egg has a tangent space.

Lemma 4.1. *If a weak egg $\mathcal{E}$ in* $\mathrm{PG}(2n + m - 1, q)$ *is good at an element E, then there exists a unique $(n + m - 1)$-space T, such that $T \cap \mathcal{E} = \{E\}$.*

Proof. Consider a $(n + m - 1)$-space Σ disjoint from E. If $\mathcal{E}$ is good at E, the element E induces a partial spread $\mathcal{S} = \mathcal{E}/E$ which extends to a Desarguesian spread $\mathcal{D}$ of Σ. By following the proof of Theorem 2.2, part (ii), for both q odd and q even, the elements of $\mathcal{D} \backslash \mathcal{S}$ span a $(m-1)$-space. It is clear that the $(n + m - 1)$-space $T = \langle E, \mathcal{D} \backslash \mathcal{S} \rangle$ satisfies $T \cap \mathcal{E} = E$. □

In [8] the authors proved that every egg of $\mathrm{PG}(7, 2)$ arises from an elliptic quadric $Q^-(3, 4)$ by field reduction. Hence, in the following characterisation, when $\mathcal{E}$ is an egg in $\mathrm{PG}(4n - 1, q)$, the condition $q^n > 4$ is essentially not a restriction.

Theorem 4.2. *Suppose $n > 1$, $q^n > 4$, consider $\mathcal{E}$ a weak egg in* $\mathrm{PG}(4n-1, q)$*. Then $\mathcal{E}$ is elementary if and only if the following three properties hold:*

- *$\mathcal{E}$ is good at an element E,*
- *there exists a $(3n-1)$-space, disjoint from E, with at least 5 elements E_1, E_2, E_3, E_4, E_5 of $\mathcal{E}$,*
- *all pseudo-ovals of $\mathcal{E}$ containing $\{E, E_1\}$, $\{E, E_2\}$ or $\{E, E_3\}$ are elementary.*

Proof. Clearly, if an egg is elementary, the statement is valid.

For the converse, consider the $(3n-1)$-space Π containing 5 elements E_1, E_2, E_3, E_4, E_5 of $\mathcal{E}$, but not the element E. As $\mathcal{E}$ is good at E, the element E induces a partial spread which extends to a Desarguesian $(n-1)$-spread $\mathcal{D}_0$ in Π, which contains E_i, $i = 1, \ldots, 5$.

By Lemma 4.1, there exists a unique $(3n-1)$-space T, such that $T \cap \mathcal{E} = \{E\}$. When $\mathcal{E}$ is an egg, this space corresponds to the tangent space T_E.

Consider the two $(n-1)$-spaces $F = \langle E_1, E_5\rangle \cap \langle E_2, E_4\rangle$ and $F' = \langle E_1, E_5\rangle \cap \langle E_3, E_4\rangle$. Both F and F' are contained in $\mathcal{D}_0$, but at most one of them can be contained in the $(2n-1)$-space $\Pi \cap T$. Suppose F is not contained in T (note that this choice has no further impact as E_2 and E_3 play the same role). This implies that the $(2n-1)$-space $\langle E, F\rangle$ contains an element $E_6 \in \mathcal{E}\setminus\{E\}$. By Theorem 3.3, there exists a unique Desarguesian spread $\mathcal{D}$ containing E, E_6 and all elements of $\mathcal{D}_0$. We will prove that $\mathcal{E}$ is contained in $\mathcal{D}$.

The $(3n-1)$-space $\langle E, E_1, E_5\rangle$ intersect $\mathcal{E}$ in a pseudo-oval $\mathcal{O}_1$, and the $(3n-1)$-space $\langle E, E_2, E_4\rangle$ intersect $\mathcal{E}$ in a pseudo-oval $\mathcal{O}_2$. Clearly, $\mathcal{O}_1$ and $\mathcal{O}_2$ both contain E_6.

By assumption, $\mathcal{O}_1$ and $\mathcal{O}_2$ are elementary pseudo-ovals. The Desarguesian $(n-1)$-spread in $\langle E, E_1, E_5\rangle$ containing $\mathcal{O}_1$ contains E, E_6 and the q^n+1 elements of $\mathcal{D}_0 \cap \langle E_1, E_5\rangle$. It follows that this Desarguesian spread is contained in $\mathcal{D}$, hence $\mathcal{O}_1$ is contained in $\mathcal{D}$. Analogously, the pseudo-oval $\mathcal{O}_2$ is also contained in $\mathcal{D}$.

There are $q^n - 2$ pseudo-ovals $\mathcal{O}$ of $\mathcal{E}$, containing $\{E, E_3\}$, but not E_6, such that the $(3n-1)$-space $\langle \mathcal{O}\rangle$ does not contain the $(n-1)$-space $T \cap \langle \mathcal{O}_1\rangle$, nor the $(n-1)$-space $T \cap \langle \mathcal{O}_2\rangle$. Take such an oval $\mathcal{O}$, then there is an element E_7 of $\mathcal{E} \setminus \{E\}$ contained in $\langle \mathcal{O}\rangle \cap \langle \mathcal{O}_1\rangle$, hence, $E_7 \in \mathcal{O} \cap \mathcal{O}_1$. Likewise, there is an element E_8 of $\mathcal{E} \setminus \{E\}$ contained in $\mathcal{O} \cap \mathcal{O}_2$.

By assumption, $\mathcal{O}$ is elementary; let $\mathcal{S}_{\mathcal{O}}$ be the Desarguesian $(n-1)$-spread containing $\mathcal{O}$. As E_7 and E_8 are contained in $\mathcal{D}$, the Desarguesian spread $\mathcal{D}$ intersects $\langle E_7, E_8\rangle$ in a Desarguesian spread. Let P be an element of $\mathcal{D} \cap \langle E_7, E_8\rangle$, not contained in T, then $\langle E, P\rangle$ meets Π in an element of $\mathcal{D}$, and hence, $\langle E, P\rangle$ contains an element P' of $\mathcal{E} \setminus E$. As $\langle E, P\rangle$ is contained in $\langle \mathcal{O}\rangle$, P' is an element of $\mathcal{O}$, and hence also of $\mathcal{S}_{\mathcal{O}}$. Since P', E, E_7, E_8 are contained in $\mathcal{S}_{\mathcal{O}}$, the element $P = \langle E, P'\rangle \cap \langle E_7, E_8\rangle$ is an element of $\mathcal{S}_{\mathcal{O}}$. This implies that $\mathcal{D} \cap \langle E_7, E_8\rangle$ and $\mathcal{S}_{\mathcal{O}}$ have at least q^n elements in common, which implies in turn that they have all their elements in common. We conclude that $\mathcal{S}_{\mathcal{O}}$ contains E, E_3 and the $q^n + 1$ elements of $\mathcal{D} \cap \langle E_7, E_8\rangle$, hence $\mathcal{S}_{\mathcal{O}}$ and thus all elements of $\mathcal{O}$ are contained in $\mathcal{D}$.

Now, consider an element $E_9 \in \mathcal{E}$, not contained in $\mathcal{O}_1$, $\mathcal{O}_2$ or any of the previously considered $q^n - 2$ pseudo-ovals $\mathcal{O}$. Look at the pseudo-oval $\mathcal{O}' = \langle E, E_1, E_9\rangle \cap \mathcal{E}$ and the pseudo-oval $\mathcal{O}'' = \langle E, E_2, E_9\rangle \cap \mathcal{E}$. At least one of them does not contain E_3. Suppose $\mathcal{O}'$ does not contain E_3 (the proof goes analogously if $\mathcal{O}''$ does not contain E_3). For at most

one of the $q^n - 2$ pseudo-ovals $\mathcal{O}$ containing $\{E, E_3\}$ we have $\langle \mathcal{O} \rangle \cap \langle \mathcal{O}'' \rangle \in T$. Hence, since $q^n - 2 \geq 3$, we can find two distinct elementary pseudo-ovals containing $\{E, E_3\}$ that are contained in $\mathcal{D}$ and have an element E_{10} and E_{11} respectively in common with $\mathcal{O}'$.

Let $\mathcal{S}_{\mathcal{O}'}$ be the Desarguesian $(n-1)$-spread containing $\mathcal{O}'$. As E_{10} and E_{11} are elements of $\mathcal{D}$ the same argument as before shows that all but one element of the Desarguesian spread $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$ can be written as the intersection of $\langle E, P'' \rangle$ with $\langle E_{10}, E_{11} \rangle$ for some P'' in $\mathcal{O}'$. It follows that $\mathcal{S}_{\mathcal{O}'}$ contains E, E_1 and the q^n+1 elements of $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$, hence, that $\mathcal{S}_{\mathcal{O}'}$ is contained in $\mathcal{D}$. In particular, the element E_9 is contained in $\mathcal{D}$, which implies that $\mathcal{E} \subset \mathcal{D}$ and so that $\mathcal{E}$ is elementary and more specifically, a field reduced ovoid. $\square$

When $\mathcal{E}$ is good at E and q is odd, by Corollary 2.3 all pseudo-ovals of $\mathcal{E}$ containing E are pseudo-conics; we use this to obtain the following corollary. The same statement, where $\mathcal{E}$ is an egg, was proven in [7, Theorem 3.2] using coordinates. For $\mathcal{E}$ an egg, this was also shown in [16, Theorem 5.2.3] where a different proof was obtained independently, relying on a technical theorem concerning the $\mathbb{F}_{q^n}$-extension of the egg elements. We have now obtained a direct geometric proof.

Corollary 4.3. *A weak egg $\mathcal{E}$ of* $\mathrm{PG}(4n-1, q)$*, q odd, $n > 1$, is classical if and only if it is good at an element E and there exists a* $(3n-1)$*-space, not containing E, with at least* 5 *elements of $\mathcal{E}$.*

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